A STUDY RELATED TO METRO DOMINATION IN GRAPHS

Synopsis of the PhD Thesis
(***Faculty of Science***)
For the Award of the Degree of

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in
MATHEMATICS

by
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1 Introduction

Graph theory as a mathematical discipline was initiated by the renowned Swiss mathematician Leonhard Euler (1707 - 1783) in his famous discussion of the Königsberg Bridge problem entitled "The solution of a problem relating to the geometry of position". Discrete Mathematics encompasses the study of graph theory, which is a revolutionary field of mathematics and computer science that addresses the topic of structures (called graphs) made up of a collection of nodes (called vertices) that are joined by lines (called edges).The impact of graph theory is significant in the disciplines of physics, computer science,chemistry, communication technology, electrical and civil engineering, economics, genetics, sociology as well as many other subjects. The theory of domination has recently been the focus of graph theory research activity due to a variety of different factors that may be developed from the fundamental notion of domination.

Domination in graphs has applications to several fields. Domination arises in facility location problems, where the number of facilities (e.g., hospitals, fire stations) is fixed and one attempts to minimize the distance that a person needs to travel to get to the closest facility. A similar problem occurs when the maximum distance to a fality is fixed and one attempts to minimize the number of facilities necessary so that everyone is serviced. Concepts from domination also appear in problems involving finding sets of representatives, in monitoring communication or electrical networks, and in land surveying (e.g., minimizing the number of places a surveyor must stand in order to take height measurements for an entire region).

1.1 Definitions and Notations

A Graph G consists of a pair $(V(G), E(G))$, where $V(G)$ is a non-empty finite set whose elements are called *vertices* and $E(G)$ is a set of unordered pair of distinct vertices in $V(G)$. The elements of $E(G)$ are called *edges* of the graph G. The cardinalities of sets $V(G)$ and $E(G)$ are called *order* and *size* of G, respectively.

If u and v are any two vertices of G, then the number of edges in the shortest path connecting them is said to be their *distance* and is denoted by $d_G(u, v)$. The notation will be simplified to $d(u, v)$ if the graph will be clear from the context.

The *eccentricity* $e_G(v)$ or $e(v)$ of a vertex v is the distance to a farthest vertex from v. The *diameter* $d(G)$ is the maximum eccentricity of G, while the *radius* $r(G)$ is the smallest eccentricity of G.

The *open neighborhood* of a vertex u in G, $N(u)$, is the set of all vertices adjacent to u. For basic definitions and notations we refer [**?**, 12]. The metric dimension of a graph $G = (V, E)$ is the cardinality of a minimal subset S of V such that for each pair of vertices u, v of V there is a vertex w in S such that the length of the shortest path from w to u is different from the length of a shortest path from w to v.

1.2 Metric Dimension

Graph structure can be used to study the various concepts of Navigation in space. A work place can be denoted as node in the graph, and edges denote the connections between the places. The problem of minimum number of machines (or Robots) to be placed at certain nodes to trace each and every node exactly once is a classical one. This problem can be solved by using networks where places are interconnected in which, the navigating agent moves from one node to another in the network. The places or nodes of a network where we place the machines (robots) are called landmarks. The minimum number of machines required to locate each and every node of the network is termed the metric dimension and the set of all minimum possible number of landmarks constitute metric basis. The machines, which are placed at nodes of the network know, their distances to sufficiently large set of landmarks and the positions of these machines on the network are uniquely determined. However there is neither the concept of direction nor that of visibility. Instead we shall assume that a Robot navigating on a Graph can sense the distance to a set of landmarks.

Also, for safeguard analysis of a facility, such as a fire protection, study of nuclear power plants, the facility can be modeled by a graph or network. For such applications, a vertex can represent a room, hallway, stairwell, courtyard, and etc. Each edge can connect two areas that are either physically adjacent or within sight or sound of each other. One primary function of a safeguards system is 'direction' of some object-perhaps detection of a fire, or detection of intruder, such as saboteur. Suppose we have a detection device located at a vertex and the device can supply three outputs,

There is an object at that vertex

1. There is an object at one of the vertices adjacent to that vertex, or

2. There is no object at that vertex or any adjacent vertex

It is necessary to determine a collection of vertices at which to place detection device so that if there is an object at any vertex in the graph, it can be detected, and its position uniquely identified.

In order to detect an object, which might be at any vertex in V(G), it is necessary to have a dominating set. The additional problem of uniquely identifying the location of the object requires a metric dimension (locating) feature.

These two concepts motivated for the investigation of the new graph invariants called locating domination number by P.J.Slater [81] and which is perfectly refined as a metro domination number by us in this chapter. First Slater introduced the concept of a locating set for a connected graph G [81]. He called the cardinality of a minimum locating set as the location number of G. Independently, F.Harary and R.A Melter introduced the notion of resolving set and metric dimension as well in 1976 [43]. Here the metric dimension of a graph G is also called a locating number of G. Applications of resolving sets arise in various areas like robot navigation [59] and network discovery [78].

Definition 1.1. *Let G be a connected graph of order n and let $S = \{u_1, u_2, \ldots, u_k\}$ be an ordered set of vertices of G. For a vertex w of G, the k-vector denoted by*

$$\Gamma(w/S) = (d(w, u_1), d(w, u_2), \ldots, d(w, u_k)),$$

is the representation of w with respect to S. .If for every pair v, w of vertices of G,

$$\Gamma(w/S) \neq \Gamma(v/S)$$

then S is called a resolving set of G. A resolving set of minimum cardinality is the metric basis and cardinality of a metric basis is the metric dimension of G denoted $\beta(G)$.

2 Proposed Thesis Outline

The thesis is organized as follows:

2.1 Chapter-1: Metro Domination Number of Graphs

Definition 2.1. *Let $D = \{u_1, u_2, \ldots, u_k\}$ be an ordered dominating set G and let w be a vertex of G. The representation $\Gamma(w/D)$ of w respect to D is the k-tuple $((d(w, u_1), (d(w, u_2), \ldots,$ $(d(w, u_k))$. If distinct vertices of G have distinct co-ordinates or representation with respect to D, then D is called a metro domination of G or simply an MD-set.*

The minimum cardinality of a minimal MD-set of G is called the lower metro domination number or simply a metro domination number of G and is denoted by $\gamma_\beta(G)$.

Bounds in Terms of Order and Diameter

Observation 2.2. *Let S be a MD set of minimum cardinality m of a graph G and $\Gamma(a/S) = (a_1.a_2, \ldots, a_m)$ be the code generated by S to the vertex a of G. Then the following hold:*

- *If $a_j = 0$, then $a_i \neq 0$ for any $i \neq j$ and $a \in S$.*
- *If $a_j \neq 0$ for any j, $1 \leq j \leq m$, then $a_i = 1$ for exactly one i and $a \in V - S$.*
- *If $a_j = b_j$ for all j, $1 \leq j \leq m$, then $a = b$ (the vertices a and b are identical.*

Theorem 2.3. *Let G be a connected graph of order n and diameter d. If $\gamma_\beta(G) = m$, then*

$$m \leq n \leq m(d-1)^{m-1} + m.$$

Lemma 2.4. *Let G be a connected graph with diameter d.*

1. *If $d = 1$, then $\gamma_\beta(G) = \beta(G)$.*
2. *If $d = 2$, then $\beta(G) \leq \gamma_\beta(G) \leq \beta(G) + 1$.*

Theorem 2.5. *Let G be a graph of order n and diameter $d \geq 3$. Then,*

$$\gamma_\beta(G) + \left\lceil \frac{2\,d}{3} \right\rceil \leq n \leq \gamma_\beta(G) + \gamma_\beta(G)\ .\ 3^{\gamma_\beta(G)-1}.$$

Theorem 2.6. *Let G be a connected graph of order n and $d \geq 3$. Then,*

$$\gamma_\beta(G) \leq n - d + \left\lfloor \frac{d}{3} \right\rfloor$$

2.1.1 Metro Domination in Circulant Graphs and Harary Graphs

Definition 2.7. *[38] A circulant graph denoted by $C_n(1, 2, \ldots, j)$, $1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$ has a vertex set $V = \{1, 2, \ldots, n\}$ and an edge set $E = \{(i, j) : |j - i| \equiv m(\text{mod } n), m \in \{1, 2, \ldots, j\}\}$. It is clear that $C_n(1)$ is an undirected cycle which is a subgraph of $C_n(1, 2, \ldots, j)$, for every $j, 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$, sometimes known as the outer cycle.*

In this chapter following are the results obtained:

Metro Domination Number of Circulant Graph $C_n(1, 3)$

Lemma 2.8. *In the circulant graph $C_n(1, 3)$, a vertex v_i dominates $v_{i-3}, v_{i-1}, v_i, v_{i+1}$ and v_{i+3}.*

Lemma 2.9. *For any positive integer $n \geq 6$, $\gamma(C_n(1, 3)) \geq \lceil \frac{n}{5} \rceil$*

Definition 2.10. *Consider a set S with two or more vertices of the graph G and let v_i and v_j be two distinct vertices of S. Further, let P and P' denote two distinct $v_i v_j$-paths in G. If either P or P', say P contains only two vertices of S namely v_i and v_j, then we refer to v_i and v_j as neighboring vertices of S. The the set of all the vertices of $P - \{v_i, v_j\}$ is called a gap of S determined by v_i and v_j and denoted by $\eta_S(v_i, v_j)$ if the following hold*

1. *v_i, v_j are neighboring vertices in S and*
2. *No vertex in S is adjacent to any vertex in $P - \{v_i, v_j\}$.*

Definition 2.11. *The minimum number of vertices in the gap is called the order of the gap, denoted by $o(\eta_S(v_i, v_j))$.*

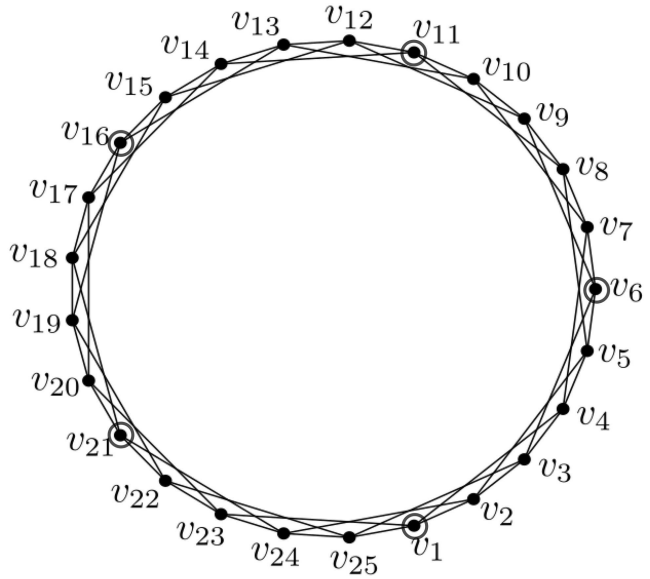

Figure 1

Lemma 2.12. *If* $n \equiv 0, 1, 2, 3(mod\ 5)$*, then* $\gamma(C_n(1,3)) = k = \lceil \frac{n}{5} \rceil$.

Lemma 2.13. *If* $n \equiv 4(mod\ 5)$*, then* $|S| \geq \lceil \frac{n}{5} \rceil + 1$.

Theorem 2.14. *For* $n \geq 22 \quad (n \neq 26)$,

$$\gamma_\beta(C_n(1,3)) = \begin{cases} \left\lceil \frac{n}{5} \right\rceil, & \text{if } n \equiv 0, 1, 2, 3(mod\ 5) \\ \left\lceil \frac{n}{5} \right\rceil + 1, & \text{if } n \equiv 4(mod\ 5) \end{cases}$$

Metro Domination of Harary Graphs

Harary graph $H_{r,n}$ is an r regular graph with $V(H_{r,n}) = \{v_1, v_2, \ldots, v_n\}$ of order n . For some integer $k \leq \frac{(n-1)}{2}$ if r is even, then $r = 2k \leq n-1$. We join v_i to $v_{i+1}, v_{i+2}, \ldots, v_{i+k}$ and to $v_{i-k}, v_{i-2}, \ldots, v_{i-k}$ for each $i(1 \leq i \leq n)$. Each vertex v_i is adjacent to the k vertices that immediately follow v_i and the k vertices that immediately precede v_i, if the vertices $v_1, v_2, \ldots, v_n$ are arranged in clockwise direction. Thus $H_{r,n}$ is r- regular.

Lemma 2.15. *For any positive integer,* $\gamma(H_{4,n}) \geq \left\lceil \frac{n}{5} \right\rceil$.

Theorem 2.16. *For* $n \geq 4$, $\gamma_\beta(H_{4,n}) \geq 3$

Theorem 2.17. *For* $n \geq 4$, $\gamma_\beta(H_{4,n}) = \begin{cases} 3, & \text{if } n = 4, 6, 7, 8, 11 \\ 4, & \text{if } n = 5, 9, 10, 12, 13, 14 \\ \left\lceil \frac{n}{5} \right\rceil, & \text{if } n \geq 15 \end{cases}$

2.2 Chapter-2: Unique Metro Domination Number of Graphs

In this chapter, concept known as Unique metro domination number of graphs is defined and further, find UMD when $G = C_n(1, 2, \ldots, k)$

Definition 2.18. *A metro dominating set* $\mathcal{D}$ *of a graph* $G(V, E)$ *is called a unique metro dominating set (in short, an UMD-set) if*

$$|N(v) \cap \mathcal{D}| = 1$$

for each vertex $v \in V - \mathcal{D}$

Definition 2.19. *A* UMD*-set of* G *is called a minimal unique metro domination number of* G *if none of its proper subsets is a UMD-set.*

Definition 2.20. *The minimum cardinality of a minimal UMD-set of a graph* G *is called a unique metro domination number and is denoted by* $\gamma_{u\beta}(G)$.

In this chapter following are the results obtained.

Restriction on degrees of the vertices with prescribed UMD number

Theorem 2.21. *Let $\mathcal{D}$ be a UMD set of G, if $\gamma_{\mu\beta}(G) = m$, then $deg(u) \leq n - m$ for $u \in V - \mathcal{D}$ and $\sum_{u\in\mathcal{D}} deg(u) \leq n + m^2 - 2m$.*

2.2.1 Unique Metro Domination in Generalised Circulant Graphs

The generalised circulant graphs are an important class of topological structures on interconnection networks. Widely used for decades in the telecommunication network. There are many good properties in these graphs such as their symmetry, fault tolerance, routing capabilities and so on. In this section, we study unique metro domination for generalised circulant graphs and in particular for $k = 2, 3$.

Definition 2.22. *Take $C_n, n \geq 2k$. If $k < i \leq n - k$, join v_i to $v_{i-2}, v_{i-3}, \ldots v_{i-k}$ and $v_{i+2}, \ldots, v_{i+k}$. If $i \leq k$, then join v_i to $v_j, j < i - 1$ and to $v_{i+2}, \ldots, v_{i+k}$. Similarly if $i > n - k$, then join v_i to $v_{i-2}, v_{i-3}, \ldots, v_{i-k}$ and all $v_j, j > i+1$ and join v_i to v_l for $l \leq i+(n+k)(mod\, n)$. The resulting graph is called $C_n(1, 2, 3, \ldots, k)$.*

Results If $k < i \leq n - k$, then v_i dominates $v_i, v_{i-1}, v_{i-2}, \ldots, v_{i-k}, v_{i+1}, v_{i+2}, \ldots v_{i+k}$. If $|i - j| \leq 2k + 1$, then vertex $v_{i+1}, v_{i+2}, \ldots v_{j-1}$ are dominated by v_i and v_j. A vertex in $C_n(1, 2, 3, \ldots, k)$ can dominate a maximum of $2k + 1$ vertices.

Lemma 2.23. *If D is a minimal dominating set then $|D| \geq \left\lceil \dfrac{n}{2k+1} \right\rceil$.*

Lemma 2.24. *Let D be a dominating set for $C_n(1, 2, 3, \ldots, k)$, then the gap with respect to the outer cycle C_n between any two vertices in D is of order less than $2k + 1$.*

Lemma 2.25. *If $n = (2k + 1)p$, then $\gamma(C_n(1, 2, \ldots, k)) = p = \left\lceil \dfrac{n}{2k+1} \right\rceil$.*

Lemma 2.26. *If $n = (2k+1)p$, then $D = \{v_{k+1}, v_{3k+2}, \ldots, v_{(2k+1)p-k}\}$ is a unique dominating set in $C_n(1, 2, \ldots, k)$. If $n \neq (2k + 1)p$ then $D = V$ is a unique dominating set of minimum cardinality.*

Lemma 2.27. *In $C_n(1, 2, \ldots, k)$, if $n = (2k + 1)(1 + 2kt), t \in \mathbb{N}$, then there exists a set $\{v_{j+1}, v_{j+2}, \ldots, v_{j+2k}\}$ of $2k$ vertices, where $j = \dfrac{n-(2k+1)}{2} + 1$, having the largest distance $\left\lceil \dfrac{j}{k} \right\rceil$ from vertex v_1.*

$C_{n_s}(1, 2, \ldots, k)$ will have $2ks + s$ more vertices and hence has s more spans of vertices of equal distance's. Thus $C_{n_s}(1, 2, \ldots, k)$ has $m + s$ spans and s vertices other than vertex v_1. These s vertices will have a distance of $m + s + 1$ from vertex v_1.This leads to,

Lemma 2.28. *There are s vertices, $1 \leq s < 2k$, such that distance from v_1 to them is $m + s + 1$.*

This in turn leads to,

Lemma 2.29. *For any* $m \in \mathbb{N}$*, there exists* $n \in \mathbb{N}$ *such that* $m = max\{d(u,v)|u,v \in C_n(1,2,\ldots,k)\}$.

Theorem 2.30. $\gamma_{\mu\beta}(C_n(1,2,\ldots,k)) = \begin{cases} \dfrac{n}{2k+1}, & \text{where } n = 2k^2+3k+1, k \in \mathbb{N} \\ n, & \text{for all other } n \end{cases}$

The following results are for $k = 2, 3$, in particular.

Unique Metro Domination of $C_n(1,2)$

Consider $C_n(1,2)$, $n \geq 4$ labelled by $v_1, v_2, \ldots, v_n$ in an anticlockwise direction. Join v_i to v_{i+2} for $1 \leq i \leq n-2, v_{n-1}$ to v_1 and v_n to v_2. The resulting graph is denoted by $C_n(1,2)$.

Lemma 2.31. *For any positive integer n,* $\gamma(C_n(1,2)) \geq \lceil \frac{n}{5} \rceil$.

Lemma 2.32. *If n=5k , then* $\gamma(C_n(1,2)) = k = \lceil \frac{n}{5} \rceil$.

If $n = 5k+1$, then by Lemma 2.31, $|D| > k$. If $v_i \in D$, then D contains $v_i, v_{i+5}, v_{i+10}, \ldots, v_{i+n-6}$. The gap between v_{i+n-6} and v_i with respect to the outer cycle C_n is of order 5. Hence we include one more vertex into D. Any one of $v_{i+n-5}, v_{i+n-4}, v_{i+n-3}, v_{i+n-2}, v_{i+n-1}$ can be included in D. Thus $|D| = k+1$ and $\frac{n}{5} < |D| < \frac{n}{5} + 1$.

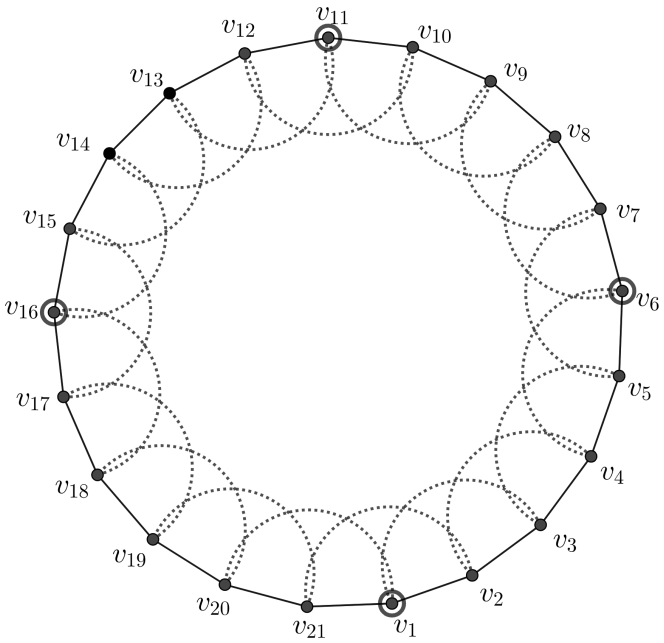

Figure 2: $C_{21}(1,2)$

Now consider the cases, $n = 5k+2, 5k+3, 5k+4$. In these cases D contains v_i, v_{i+5}, v_{i+10}, ..., v_{i+n-6} and the gap between v_{i+n-6} and v_i with respect to the outer cycle C_n is of

order greater than 4. Therefore one more vertex is to be included in D. Thus, $|D| = k + 1$ and $\frac{n}{5} < |D| < \frac{n}{5} + 1$. Hence we have $\gamma(C_n(1,2)) = \lceil \frac{n}{5} \rceil$.

Lemma 2.33. *Let D be a unique dominating set for $C_n(1,2)$. If v_i and $v_j \in D$ have a gap of the order less than 4 with respect to C_n then $D = V$.*

Lemma 2.34. *Let D be a unique dominating set for $C_n(1,2)$, then the gap with respect to C_n between any two vertices in D is of the order less than 5.*

Lemma 2.35. *If all the gaps are of order 4, then D is a unique dominating set.*

Therefore $D = \{v_{5p+1} | p \in \mathbb{N}, 0 \leq p < k\}$ is a UMD set and hence we have

$$\gamma_{\mu\beta}(C_n(1,2)) = \begin{cases} \frac{n}{5}, & \text{where } n = 5k, k \in \mathbb{N}, k \geq 3 \\ n, & \text{for all other } n \end{cases}$$

Unique Metro Domination of $C_n(1,2,3)$
Consider C_n, $n \geq 6$ labeled by $v_1, v_2, \ldots, v_n$ in anticlockwise direction. Join v_i to v_{i+2} and v_{i+3} for $1 \leq i \leq n-3, v_{n-1}$ to v_1 and v_2 and from v_n to v_2 and v_3. The resulting graph is denoted by $C_n(1,2,3)$.

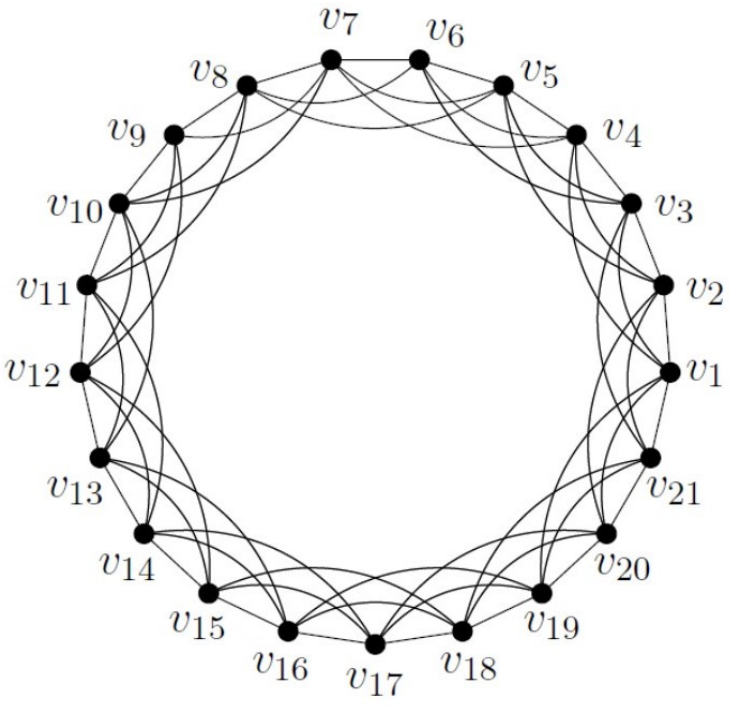

Figure 3: $C_{21}(1,2,3)$

Lemma 2.36. *For any positive integer n, $\gamma(C_n(1,2,3)) \geq \lceil \frac{n}{7} \rceil$.*

Lemma 2.37. *If n=7k , then* $\gamma(C_n(1,2,3)) = k = \lceil \frac{n}{7} \rceil$.

If $n = 7k+1$, then by Lemma 2.36 , $|D| > k$. If $v_i \in D$, then D contains $v_i, v_{i+7}, v_{i+14}, \ldots, v_{i+n-8}$. The gap between v_{i+n-8} and v_i with respect to C_n is of order 7. Hence we include one more vertex into D. Any one of $v_{i+n-7}, v_{i+n-6}, v_{i+n-5}, v_{i+n-4}, v_{i+n-3}, v_{i+n-2}, v_{i+n-1}$ can be included in D. Thus $|D| = k+1$ and $\frac{n}{7} < |D| < \frac{n}{7} + 1$.

Now consider the cases, $n = 7k+2, 7k+3, 7k+4, 7k+5, 7k+6$. In these cases D contains v_i, v_{i+7}, v_{i+14}, ..., v_{i+n-8} and the gap between v_{i+n-8} and v_i with respect to C_n is of order greater than 6. Therefore one more vertex is to be included in D. Thus, $|D| = k+1$ and $\frac{n}{7} < |D| < \frac{n}{7} + 1$. Hence we have $\gamma(C_n(1,2,3)) = \lceil \frac{n}{7} \rceil$.

Lemma 2.38. *Let D be a dominating set for* $C_n(1,2,3)$*, then the gap with respect to* C_n *between any two vertices in D is of the order less than 7.*

Lemma 2.39. *Let D be a unique dominating set for* $C_n(1,2,3)$*. If* v_i *and* $v_j \in D$ *have a gap of order less than 6 with respect to* C_n *then* $D = V$*.*

If gap between v_i and v_j with respect to C_n is of order 6, then $j = i + 7$ and $v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}$ are uniquely dominated by v_i and v_j.

Lemma 2.40. *If all gaps are of order 6, then D is a unique dominating set.*

Therefore $D = \{v_{7p+1} | p \in \mathbb{N}, 0 \leq p < k\}$ is a unique metro dominating set and hence we have

$$\gamma_{\mu\beta}(C_n(1,2,3)) = \begin{cases} \frac{n}{7}, & \text{where } n = 7k, k \in \mathbb{N}, k \geq 4 \\ n, & \text{for all other } n \end{cases}$$

2.3 Chapter-3: Unique Metro Domination Number of Power of Paths

The $k-$th power of a path is a graph obtained from a path and joining all pair of vertices of the path with distance less than k. Power graphs are used to increase the connectivity of an existing network so it is necessary to study the metric dimensions and its domination. In this chapter, we give a characterisation of the unique metro domination of power of paths. In this chapter following are the results obtained.

Results

If $k < i \leq n-k$, join v_i to $v_{i-2}, v_{i-3}, \ldots v_{i-k}$ and $v_{i+2}, \ldots, v_{i+k}$. If $i > n-k$, then join v_i to $v_{i-2}, v_{i-3}, \ldots, v_{i-k}$ and all $v_j, j > i+1$. Similarly if $i \leq k$, then join v_i to $v_j, j < i-1$ and to $v_{i+2}, \ldots, v_{i+k}$. The resulting graph is called P_n^k.

If $k < i \leq n-k$, then v_i dominates $v_i, v_{i-1}, v_{i-2}, \ldots, v_{i-k}, v_{i+1}, v_{i+2}, \ldots v_{i+k}$. If $|i-j| \leq 2k+1$, then vertex $v_{i+1}, v_{i+2}, \ldots v_{j-1}$ are dominated by v_i and v_j.

Lemma 2.41. *If D is a minimal dominating set then* $|D| \geq \left\lceil \frac{n}{2k+1} \right\rceil$.

Lemma 2.42. *When* $n = k+1+(2k+1)p, p \in \mathbb{N}$, *the domination number* $\gamma(P_n^k) = \frac{n}{2k+1} = \left\lceil \frac{n}{2k+1} \right\rceil$.

Lemma 2.43. *If* $n = 2k^2+2k+1$, *then* $D = \{v_{k+1}, v_{3k+2}, \ldots\}$ *is a unique metro dominating set.*

Also if $n \geq 2k^2+2k+1$ and $(2k+1)p - 2k \leq n < (2k+1)p-k$, then $D = \{v_1, v_{2k+2}, v_{4k+3}, \ldots v_{(2k+1)p-2k}\}$ is a UMD set.
In any case $|D| = p = \left\lceil \frac{n}{2k+1} \right\rceil$. Hence we have,

$$\gamma_{\mu\beta}(P_n^k) = \begin{cases} \left\lceil \frac{n}{2k+1} \right\rceil, & \text{for } n \geq 2k^2+k+1 \\ n, & \text{for } n < 2k^2+k+1 \end{cases}$$

The following results for $k = 2, 3$ in particular are investigated.
Unique Metro Domination in P_n^2
Consider P_n, $n \geq 3$. Join v_i to v_{i+2} for $1 \leq i \leq n-2$. The resulting graph is denoted by P_n^2.

Lemma 2.44. *For any positive integer n,* $\gamma(P_n^2) \geq \lceil \frac{n}{5} \rceil$.

Lemma 2.45. *If n=5k ,* $k \in \mathbb{N}$ *then* $\gamma(P_n^2) = k = \lceil \frac{n}{5} \rceil$.

If $n = 5k+1$, $n = 5k+2$, $n = 5k+3$, the set $\{v_1, v_6, v_{11}\}$ is a resolving set and the set $D = \{v_1, v_6, v_{11}, ..., v_{5k-4}, v_{5k+1}\}$ is a UMD set. Therefore $\gamma_{\mu\beta}(P_n^2) = k+1$.
If $n = 5k+4$, $D = \{v_2, v_7, v_{12}, ..., v_{5k-3}, v_{5k+2}\}$ is a UMD set and we have $\gamma_{\mu\beta}(P_n^2) = k+1 = \lceil \frac{n}{5} \rceil$. Thus we obtain, $\gamma_{\mu\beta}(P_n^2) = \lceil \frac{n}{5} \rceil$, $\forall n \geq 11$. we have for any integer $n \geq 3$,

$$\gamma_{\mu\beta}(P_n^2) = \begin{cases} \lceil \frac{n}{5} \rceil, & \text{for } n \geq 11 \\ n, & \text{for } n < 11 \end{cases}$$

Unique Metro Domination in P_n^3
Consider P_n, $n \geq 4$. Join v_i to v_{i+2} and v_{i+3} for $1 \leq i \leq n-3$. The resulting graph is denoted by P_n^3.

Lemma 2.46. *For any positive integer n,* $\gamma_{\mu\beta}(P_n^3) \geq \lceil \frac{n}{7} \rceil$.

Lemma 2.47. *For* $n = 7k$, $\gamma(P_n^3) = \lceil \frac{n}{7} \rceil$

Lemma 2.48. *If n=7k , then* $\gamma_{\mu\beta}(P_n^3) = k = \lceil \frac{n}{7} \rceil$.

When $n = 7k+1, 7k+2, 7k+3$,and $7k+4$,$\{v_1, v_8, v_{15}, v_{22}\}$ is a resolving set and $D = \{v_1, v_8, v_{15}, ..., v_{7k-6}, v_{7k+1}\}$ is a UMD set.

When $n = 7k+5, 7k+6$, $D = \{v_4, v_{11}, v_{18}, ..., v_{7k-3}, v_{7k+4}\}$ is a UMD set. Therefore $\gamma_{\mu\beta}(P_n^3) = k+1$. In all these cases $\mid D \mid = k+1 = \lceil \frac{n}{7} \rceil$. Hence we have,

$$\gamma_{\mu\beta}(P_n^3) = \begin{cases} \lceil \frac{n}{7} \rceil, & \text{for } n \geq 22 \\ n, & \text{for } n < 22. \end{cases}$$

2.4 Chapter-4: Global Metro Domination Number of Graphs

In this chapter, we combine the concepts of global domination and metro domination to define global metro domination in graphs.

Let $G = (V, E)$ be a graph. A dominating set $\mathcal{D}$ of a graph G is called a metro dominating set if for each ordered subset $\mathcal{D} = \{v_1, v_2, v_3, ..., v_k\}$ of V, each vertex $v \in V$ can be associated with a vector of distances denoted by

$$\Gamma(v/\mathcal{D}) = (d(v_1, v), d(v_2, v), ..., d(v_k, v))$$

such that $\Gamma(v/\mathcal{D}) \neq \Gamma(u/\mathcal{D})$, for every $u, v \in V - \mathcal{D}$. If $\mathcal{D}$ is a metro dominating set of both G and its complement $\overline{G}$, then $\mathcal{D}$ is called a global metro dominating set (GMD-set) of G. The minimum cardinality of a GMD-set of G is called the global metro domination number of G and is denoted by $\gamma_{g\beta}(G)$. In this chapter we present several basic results on $\gamma_{g\beta}(G)$ and interesting problems for further investigation.

Definition 2.49. *Let $G = (V, E)$ be a graph. A subset D of V is a global metro dominating set (g.m.d set) of G if D is a metro dominating set of G and its complement $\overline{G}$. The minimum cardinality of a global metro dominating set of G is called the global metro domination number of G and is denoted by $\gamma_{g\beta}(G)$.*

Certainly, an metro dominating set S is $full - degree$ if and only if there exists a unique vertex v in $V(G) - S$ which dominates S.
Since at least two vertices are required to dominate and resolve both G and $\overline{G}$, we have $2 \leq \gamma_{g\beta}(G) \leq n$ for every graph of order $n > 1$. In particular, the equality of the lower bound is attained by $P_n (2 \leq n \leq 6)$ and $K_{r,s} (|r-s| \leq 1)$ while the upper bound is achieved by $K_n, K_{1,r}$ and $K_{r,s} (|r-s| \geq 2)$.
In this chapter following are the results obtained.

Proposition 2.50. *If G is a graph with a full-degree MD set $\mathcal{D}$ and v is the only vertex dominating $\mathcal{D}$, then the following conditions are satisfied:*

1. *the eccentricity e(v) of $G \leq 2$;*
2. *the radius $r(G) \leq 2$;*
3. *the diameter $d(G) \leq 4$.*

Corollary 2.51. *If a graph G satisfies $\gamma_\beta(\overline{G}) = \gamma_\beta(G) + 1$, then G is a connected graph with $rad(G) \leq 2$ and $diam(G) \leq 4$.*

Definition 2.52. *We mean, by a graph G satisfies $property - 1$, if*

$$d(u) + d(v) \leq 4 \quad \forall u, v \in G$$

Henceforth, if G satisfies Definition 2.52 we call it as graph G is with $property - 1$.

Global Metro Domination in Graphs with $Property - 1$

Theorem 2.53. *If G is a graph which satisfies $Property - 1$, then $\gamma_{g\beta}(G) = \left\lceil \frac{2n}{5} \right\rceil$, for $n \geq 7$.*

Global Metro Domination in Cycles

Lemma 2.54. *If D is a dominating set of C_n , then the order of the gap between any two vertices in D is at most 2.*

Lemma 2.55. *If $n = 5k$, $\gamma_{g\beta}(C_{5k}) = 2k$.*

Lemma 2.56. *When $n = 5k + 1$, $\gamma_{g\beta}(C_n) = 2k + 1$.*

Lemma 2.57. *When $n = 5k + 2$, $\gamma_{g\beta}(C_n) = 2k + 1$.*

Lemma 2.58. *When $n = 5k + 3$, $\gamma_{g\beta}(C_n) = 2k + 2$.*

Lemma 2.59. *When $n = 5k + 4$, $\gamma_{g\beta}(C_n) = 2k + 2$.*

Summarizing all these results, we have

Theorem 2.60. *For the cycle C_n ($n \geq 7$), $\gamma_{g\beta}(C_n) = \left\lceil \frac{2n}{5} \right\rceil$.*

Global Metro Domination in Paths Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of the path P_n.

A vertex of G that is of degree 1 in G is called an *end vertex* of G. The vertex adjacent to an end vertex v is called a *support vertex* of v in G.

Lemma 2.61. *For a path P_n, $n \geq 3$, either an end vertex or a support vertex is in any GMD-set.*

Lemma 2.62. *In a path P_n, $n \geq 2$, for any GMD set D of minimum cardinality, if $u, v \in D$ and there is no $w \in D$ between u and v, then $d(u, v) \leq 3$.*

Similarly the cases where $n = 5k + i, 1 \leq i \leq 4$, can be discussed and we get,

Theorem 2.63. *For the path P_n ($n \geq 7$), $\gamma_{g\beta}(P_n) = \left\lceil \frac{2n}{5} \right\rceil$.*

From theorem 2.60 and 2.63, if $d(u) + d(v) \leq 4$, $\forall u, v \in G$ then $\gamma_{g\beta}(G) = \left\lceil \frac{2n}{5} \right\rceil$, for $n \geq 7$.

Characterization of Graphs with the Property $\gamma_{g\beta}(G) = max\{\gamma_\beta(G), \gamma_\beta(\overline{G})\}$.

Theorem 2.64. *For the complete graph $K_n, (n \geq 2)$, $\gamma_{g\beta}(K_n) = n$.*

Theorem 2.65. *For the star graph $K_{1,n-1}, (n \geq 2)$, $\gamma_{g\beta}(K_{1,n-1}) = n - 1$.*

Theorem 2.66. *For the wheel graph $W_n (n \geq 7)$, $\gamma_{g\beta}(W_n) = \left\lceil \frac{(2n+1)}{5} \right\rceil$.*

Theorem 2.67. *Let G be a graph of order $n \geq 1$ which belongs to the set $\{W_n, K_n, K_{1,n-1}, K_{s,n-s}\}$, then the values of $\gamma_{g\beta}(G)$ are known and displayed in Table 1.*

G	Wn	K_n	$K_{1,n-1}$	$K_{s,n-s}$
order n	$n \geq 7$	$n \geq 2$	$n \geq 4$	$2 \leq s \leq n - s$
$\gamma_\beta(G)$	$\left\lceil \frac{2(n-1)}{5} \right\rceil$	$n - 1$	$n - 1$	$n - 2$
$\gamma_\beta(\overline{G})$	$\left\lceil \frac{2n+1}{5} \right\rceil$	n	$n - 1$	$n - 2$
$\gamma_{g\beta}(G)$	$\left\lceil \frac{2n+1}{5} \right\rceil$	n	$n - 1$	$n - 2$

Table 1: The values of $\gamma_\beta(G), \gamma_\beta(\overline{G}), \gamma_{g\beta}(G)$ for some families of graphs.

Nordhaus-Gaddum Type Bounds

Theorem 2.68. *For every nontrivial graph G of order $n \geq 2$,*

$$3 \leq \gamma_\beta(G) + \gamma_\beta(\overline{G}) \leq 2n - 1.$$

Furthermore,

- $\gamma_\beta(G) + \gamma_\beta(\overline{G}) = 3$ *if and only if* $\{G, \overline{G} = \{K_2, \overline{K_2}\}$.
- $\gamma_\beta(G) + \gamma_\beta(\overline{G}) = 2n - 1$ *if and only if* $\{G, \overline{G}\} = \{K_n, \overline{K_n}\}$

Theorem 2.69. *If G and $\overline{G}$ are both connected graphs of order $n \geq 5$,*

$$4 \leq \gamma_\beta(G) + \gamma_\beta(\overline{G}) \leq 2n - 5.$$

2.5 Chapter-5: Global Metro Domination Number of Trees

One of the important classes of graphs is trees. The importance of trees is evident from their applications in various areas, especially theoretical computer science and molecular evolution. This chapter begins with determining the MD number of a tree in relation to its domination number. In particular, we prove that $\gamma(T) = \gamma_\beta(T)$ if and only if T includes no vertex adjacent to two or more end vertices. Further we find the bounds for global metro domination number of trees in terms of clique.

Definition 2.70. *A graph is acyclic if it has no cycles. A tree T is a connected acyclic graph.*

Definition 2.71. *In a tree T, if $deg(v) = 1$ then v is a $leaf$, and the only vertex u adjacent to v is called its support vertex. A strong support vertex u has two or more leaves adjacent to it.*

For a tree T, $S(T)$ denote the set of support vertices of T, $S_s(T)$ denote the set of strong support vertices of T and the total number of leaves adjacent to a strong support vertex in T is denoted by (T). $L(T)$ is the set of all leaves of T, $R(T)$- the set of the remaining vertices of T ($R(T) = V(T) - (S(T) \cup S_s(T) \cup L(T))$

Definition 2.72. *[21] A $clique$ in a graph G is a maximal complete subgraph of G.*

Observation 2.73. *If G is a connected graph containing a strong support vertex v and the leaves $v_1, v_2, \ldots, v_k$ adjacent to v. Let D be a dominating set of G. Take any $u \in D$, $u \neq v_i, 1 \leq i \leq k$. If $d(u, v) = a$, then $d(u, v_i) = a + 1$. Then all v_i are equidistant from u. Hence every MD set of G includes all the leaves adjacent to v or all but one of the leaves v_j, $(1 \leq j \leq k)$ adjacent to v in addition to the vertex v.*

Characterization of Graphs with the Property $\gamma_\beta(T) = \gamma(T)$

In this section, we first define the trees for which $\gamma_\beta(T) = \gamma(T)$ and secondly determine the MD number of a tree in relation to its domination number.

Lemma 2.74. *Every dominating set of a tree T is an MD set if T contains no strong support vertex.*

Corollary 2.75. *If a tree T contains no strong support vertex, then $\gamma(T) = \gamma_\beta(T)$.*

Theorem 2.76. *Let T be a tree, then $\gamma(T) = \gamma_\beta(T)$ if and only if T contains no strong support vertex.*

Global Metro Domination of Trees

Theorem 2.77. *If T is a tree of order $n \geq 4$, then*

$$\gamma_\beta(T) = \gamma(T) + (T) - |S_s(T)|$$

Lemma 2.78. *If vertex $v_i \in S_S(T)$ in a tree T, then every global metro dominating set D consists of all the leaves in T or all v_i and leaves adjacent to v_i except one leaf.*

Theorem 2.79. *Every maximal clique of $\overline{T}$ is a global metro dominating set of T.*

The converse is not true.
Consider the figure 4, the set $D_1 = \{v_1, v_2, v_3, v_5, v_7, v_9, v_{11}\}$ forms a maximal clique in $\overline{T}$. Here D_1 dominates and resolves all the vertices in T and $\overline{T}$. Hence, D_1 is a GMD set of T, but $D_2 = \{v_1, v_2, v_3, v_5, v_8, v_{10}\}$ also forms a GMD set of T.

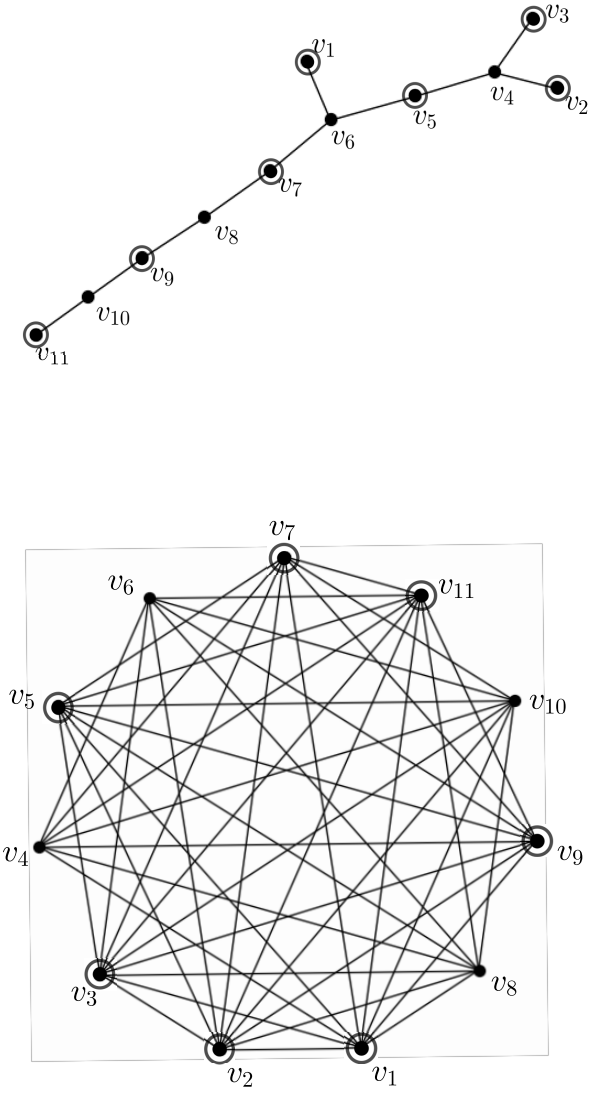

Figure 4: Tree T and its complement with $\omega(\overline{T}) = 7$

We have the bound for global metro dominating for a tree T,

$$\gamma_\beta(T) \leq \gamma_{g\beta}(T) \leq \omega(\overline{T}).$$

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3 List of Publications/ Paper presented/ Communicated

3.1 Publications in Journals/ Communicated

1. Uni-Distance Domination of Square of Paths.
   Kishori P. Narayankar, **Denzil Jason Saldanha**, John Sherra.
   International Journal of Mathematical Combinatorics. **2**, (2020), 118-120.

2. Unique Metro Domination of Cube of Paths.
   Kishori P. Narayankar, **Denzil Jason Saldanha**, John Sherra.
   International Journal of Mathematics Trends and Technology (IJMTT). **66(3)**, (2020).

3. Unique Metro Domination of Square of Cycles.
   Kishori P. Narayankar, **Denzil Jason Saldanha**, John Sherra.
   Journal of Research in Applied Mathematics. **6(1)**, (2020).

4. Unique Metro Domination of Power ofPaths.
   Kishori P. Narayankar, **Denzil Jason Saldanha**, John Sherra.
   Advances in Mathematics: Scientific Journal. **10(1)**, (2021), 311-317.

5. Global Metro Domination in Graphs.
   Kishori P. Narayankar, **Denzil Jason Saldanha**, John Sherra.
   Advances and Applications in Discrete Mathematics. **27(2)**, (2021), 233-247.

6. Unique Metro Domination of Circulant Graphs.
   Kishori P. Narayankar, **Denzil Jason Saldanha**, John Sherra.
   AIP Conf. Proceedings. (Accepted).

7. Unique Metro Domination of Generalised Circulant Graphs.
   Kishori P. Narayankar, **Denzil Jason Saldanha**, John Sherra.
   Walailak Journal of Science and Technology. (Accepted).

8. Global Metro Domination in Graphs.
   Kishori P. Narayankar, **Denzil Jason Saldanha**, John Sherra.
   (Communicated).

3.2 Papers Presented in International/National Conference

1. Unique Metro Domatic Number in Paths, (with Kishori P. Narayankar) presented in the International conference on Number Theory and Graph Theory, organized by Department of Studies in Mathematics, University of Mysore, Mysuru.

2. Unique Metro Domination in Circulant Graphs, (with Kishori P. Narayankar) presented in the International Conference on Advances in Multi-Disciplinary Sciences and Engineering Research organised by Chitkara University, Himachal Pradesh.

3. Global Metro Domination, (with Kishori P. Narayankar) presented in the International conference on Discrete Mathematics, organized by organized by Department of Mathematics, Mangalore University, Mangalore and Academy of Discrete Mathematics (ADMA).

3.3 Workshop/Conference Attended

1. Attended Five Day International Research Workshop on Graph Domination (IRWGD-21) organised by the Department of Mathematics, St Paul's College, Kalamassery in association with the Ramanujan Mathematical Society.

2. Attended Virtual International Conference on Discrete Mathematics and Combinatorics 2021, organised by the Department of Mathematics, Poornaprajna College, Udupi.

3. Attended a One Week Short Term Course on Computational Software on Matlab & Mathematica under Centre for Continuing Education, SVNIT Surat organised by Sardar Vallabhbhai National Institute of Technology, Surat, Gujarat.

4. Participated in the National level webinar on Mathematical Software Workshop organized by the Department of Applied Sciences and Humanities in Association with Universal Engineering College, Thrissur, Kerala.

5. Participated in online Workshop on Recent Trends in Domination and Graph Labelling (RTDGL 2021) organised by Department of Mathematics, Ramanujan College, University of Delhi in collaboration with Academy of Discrete Mathematics and Applications (ADMA).

6. Participated in the Webinar Applied Graph Teory- An Overview organized by the Department of Mathematics, GSSS Institute of Engineering Technology for Women, Mysuru

7. Participated in the National Webinar on Recent trends and Applications of Graph Theory, organised by the Government Degree College, Pithapuram.

8. Participated in a National webinar on Discrete Math and Computing: The aesthetic interplay organised by the Depart- ment of Mathematics, Dr. Ambedkar Institute

of Technology, Bengaluru in association with Institute of Engineering and Technology,Rammanokar Lohia Avadh University, Ayodya..

9. Participated in One Week Online National level Faculty Development Programme on The Role of Discrete Mathematics, Combinatorics Graph Theory in Science and Technology organized by the Department of Mathematics, JNN College of Engineering, Shivamogga, Karnataka, India.

10. Participated in three days international workshop on Research, Writing and Publishing in Reputed Journals organised by Centre of Excellence for Research Innovation and Entrepreneurship in association with Indian Society of Technical Education Pace chapter..

11. Attended Two Day National Level Skill Development Workshop on Scilab, by Department of Mathematics Mangalore University, Mangalore, India.

12. Attended Three Days Virtual National Workshop on Domination Theory and Topological Indices, by Department of Mathematics Mangalore University, Mangalore, India.

Candidate Signature
. (Denzil Jason Saldanha)
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Signature of The
Research Guide and Chairperson
(Dr. Kishori P. Narayankar)